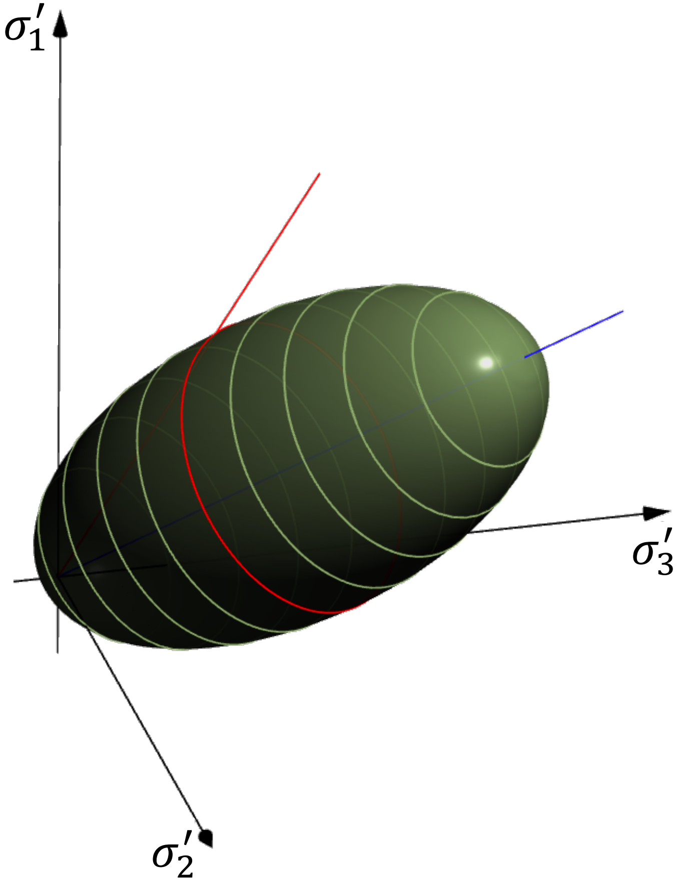


Geomechanics

LECTURE 7

MODIFIED CAM-CLAY MODEL

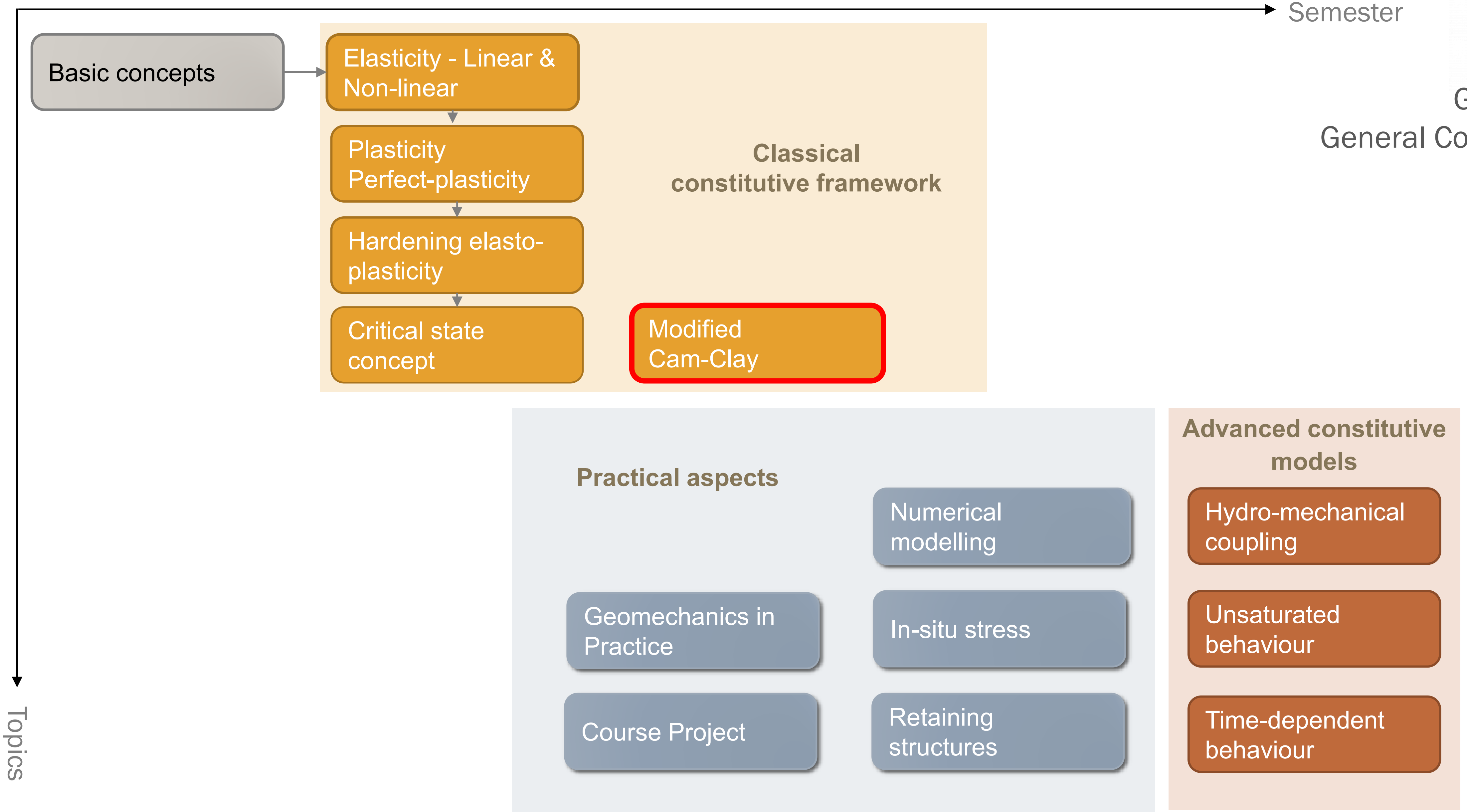
DR. ALESSIO FERRARI

Laboratory of soil mechanics – Fall 2025

Access the QUIZ

https://etc.ch/YBGR

Geomechanics
General Course overview



Semester

Topics

Basic concepts

Classical constitutive framework

Elasticity - Linear & Non-linear

Plasticity Perfect-plasticity

Hardening elasto-plasticity

Critical state concept

Modified Cam-Clay

Practical aspects

Numerical modelling

Geomechanics in Practice

In-situ stress

Course Project

Retaining structures

Advanced constitutive models

Hydro-mechanical coupling

Unsaturated behaviour

Time-dependent behaviour

Content

- Experimental evidences
- Modified Cam-Clay model
  - Elastic behaviour
  - Yield function
  - Plastic potential
  - Hardening rule
- MCC behaviours
- Conclusion

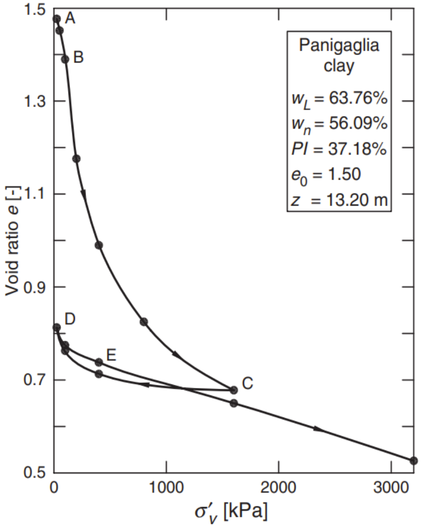
Experimental evidences

Volumetric behavior in compression

Fine-grained soils

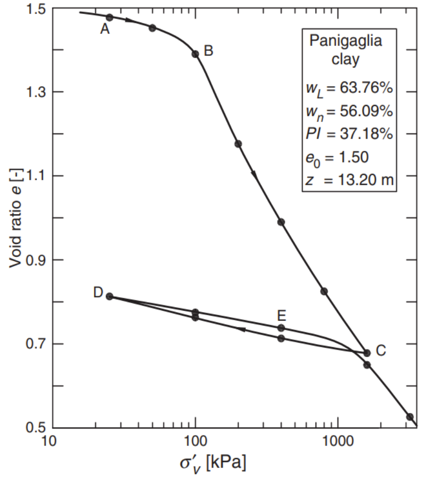
Coarse-grained soils

Natural scale



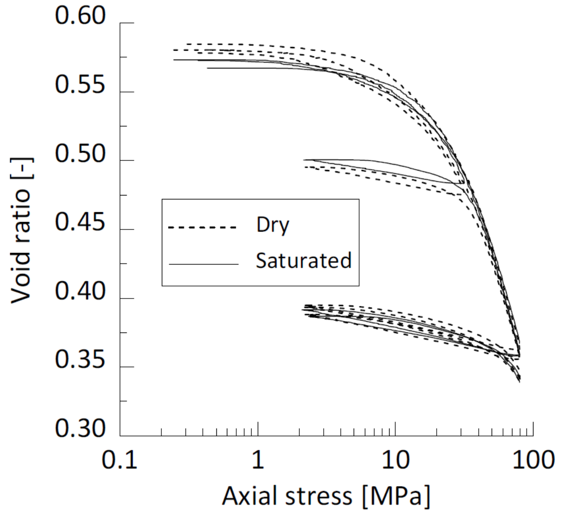
(a)

Semi-logarithmic scale



(b)

Lancellotta, 2008



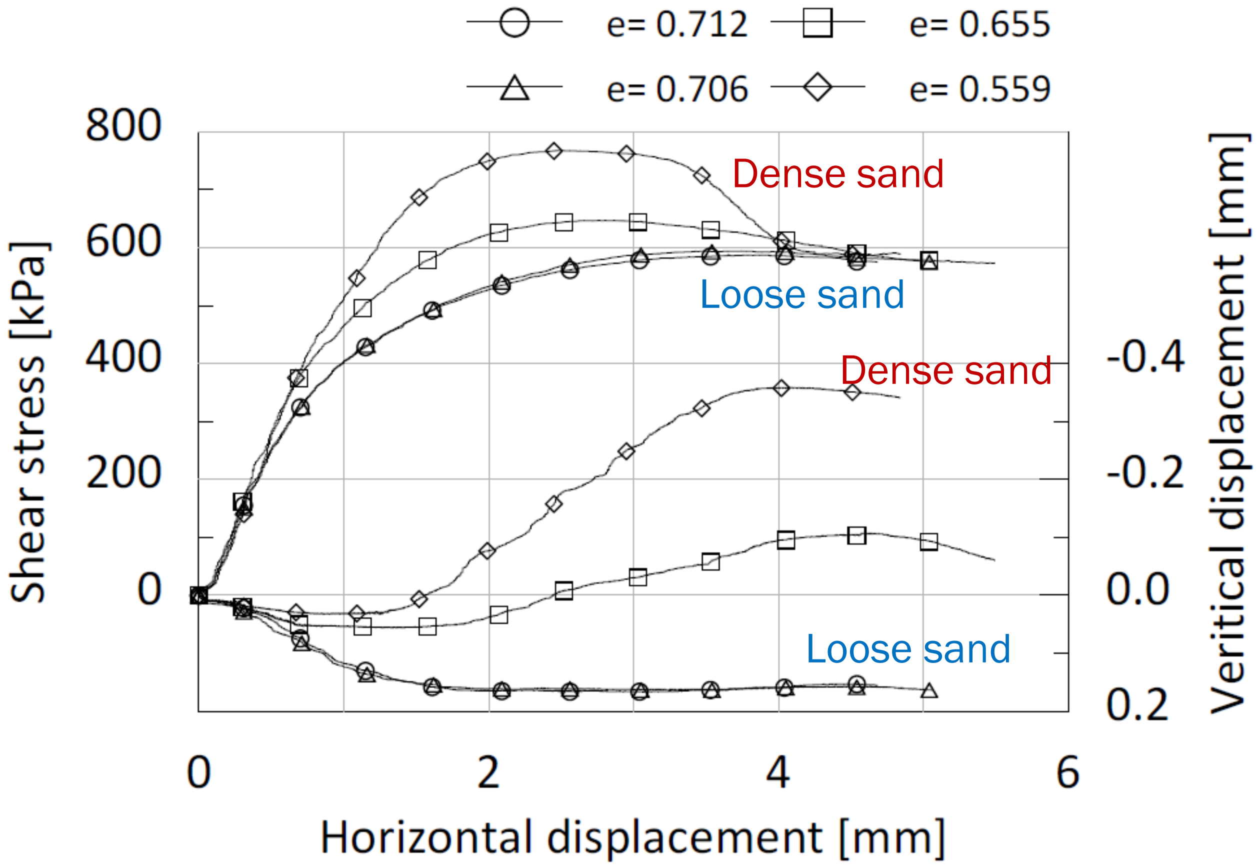
Taeheon Kim, EPFL PhD thesis, 2022

- The volumetric response in compression (both for the oedometric and the triaxial tests) is conveniently represented in semi-logarithmic plots, highlighting:
  - (i) linear trends both in the normally-consolidated and the over-consolidated states (non-linearity in reality!);
  - (ii) the change in the preconsolidation effective stress (vertical for oedometric, mean for triaxial) as the maximum load progresses.

Critical state behaviour

- Results of direct shear tests on a sand prepared at different void ratios, showing:
  i. peak and post-peak response associated with dilatant behaviour for the denser configuration;
  ii. ductile response associated with contractive behaviour for the looser configuration.

- Note that volumetric behaviour stabilises when the shear stress reaches a constant value (critical state).



Taeheon Kim, PhD thesis, 2022

Modified Cam Clay model (MCC)

ELASTIC BEHAVIOUR

YIELD FUNCTION

PLASTIC POTENTIAL

HARDENING RULE

MCC - Introduction

Originally developed at University of **Cambridge** for saturated **clay**

Original Cam-Clay – Roscoe et al. 1958

Further works on the original model yielded a modified version

Modified Cam-Clay - Schofield & Worth 1968

The model is based on:

- **Critical state concept**
- **Strain hardening elasto-plasticity**

MCC – Stress and strain framework

Recall on triaxial stress-strain

Mean effective stress: $p' = \frac{\sigma_1 + 2\sigma_3}{3} = \frac{J_1}{3}$

Volumetric strain: $\varepsilon_v = \varepsilon_1 + 2\varepsilon_3$

Deviatoric stress: $q = \sigma_1 - \sigma_3 = \sqrt{3\mathrm{J}_{2\mathrm{D}}}$

Deviatoric strain: $\varepsilon_d = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$

Ingredients of a hardening elastoplastic model

(i) elastic behaviour

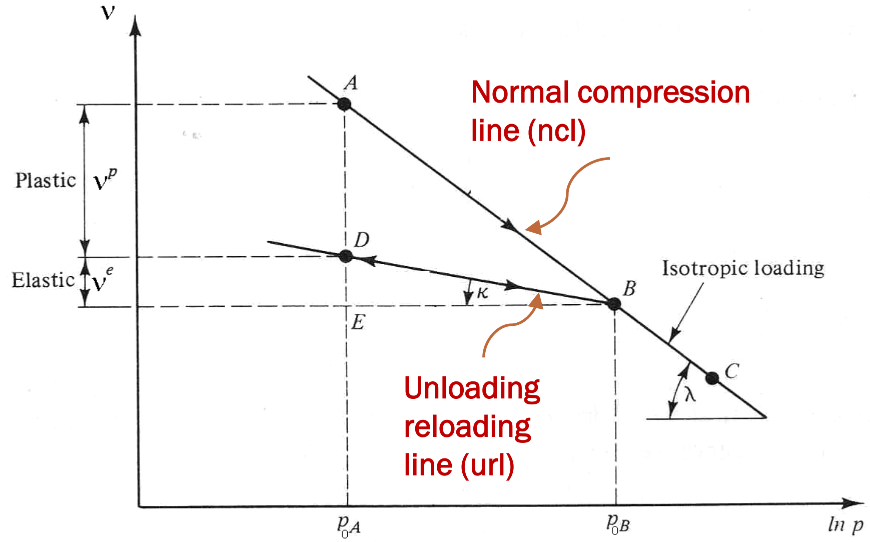
(ii) yield surface

(iii) plastic potential and flow rule

(iv) hardening rule

MCC – Elastic behaviour

Volumetric behaviour to be read on the unloading reloading line (non linear elasticity)



Normal compression line (ncl)

Unloading reloading line (url)

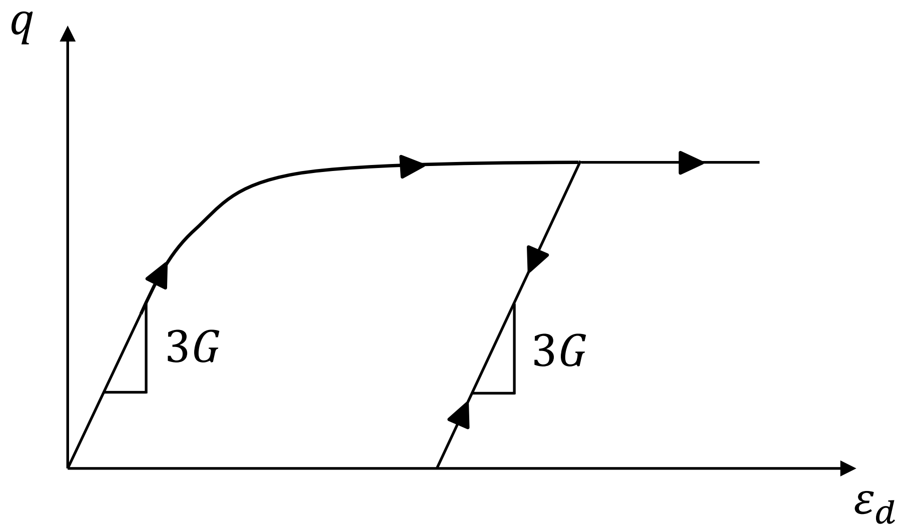
v is the **specific volume**, $v = V/V_s = 1 + e$ ➡ $\boxed{\Delta v = \Delta e}$

$$\Delta e^e = e_D - e_E = -\kappa \ln\left(\frac{p'_B}{p'_A}\right) \Rightarrow \boxed{de^e = -\kappa \frac{dp'}{p'}}$$

Recall:

$$d\varepsilon_v = -\frac{de}{1+e} = -\frac{dv}{v} \Rightarrow \boxed{d\varepsilon_v^e = -\frac{de^e}{1+e} = \frac{\kappa}{1+e}\frac{dp'}{p'} = \frac{\kappa}{v}\frac{dp'}{p'}}$$

Deviatoric behaviour - linear elasticity



$$\boxed{d\varepsilon_d^e = \frac{dq}{3G}}$$

MCC – Elastic part - Summary

In isotropic linear elasticity

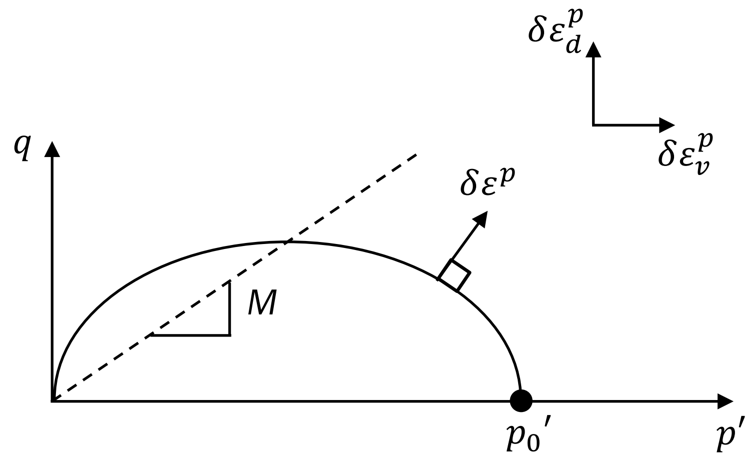
$$\begin{bmatrix} d\varepsilon_v^e \\ d\varepsilon_d^e \end{bmatrix} = \begin{bmatrix} {}^1\!/_K & 0 \\ 0 & {}^1\!/_{3G} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

In MCC

$$\begin{bmatrix} d\varepsilon_v^e \\ d\varepsilon_d^e \end{bmatrix} = \begin{bmatrix} {}^\kappa\!/_{vp'} & 0 \\ 0 & {}^1\!/_{3G} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

MCC – Yield surface

MCC in q-p' plane



Wood, 1990

$$F = q^2 - M^2[p'(p_0' - p')] = 0$$

- M is the slope of the line passing through the origin and the vertex of the ellipse (it is a critical state line)
- p_0' is the right vertex of the ellipse (geometrically). Physically it represents the maximum mean effective stress previously experienced along an isotropic compression path.

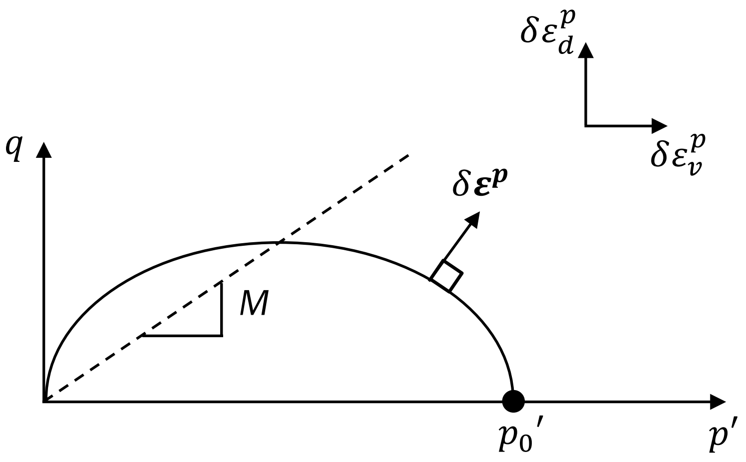
Equivalently in terms of stress ratio (η):

$$\frac{p'}{p_0'} = \frac{M^2}{M^2 + \eta^2}$$

$$\eta = \frac{q}{p'}$$

MCC – Plastic potential and flow rule

MCC in q-p' plane



Wood, 1990

Associated flow rule

$$g = F = q^2 - M^2[p'(p_0' - p')] = 0$$

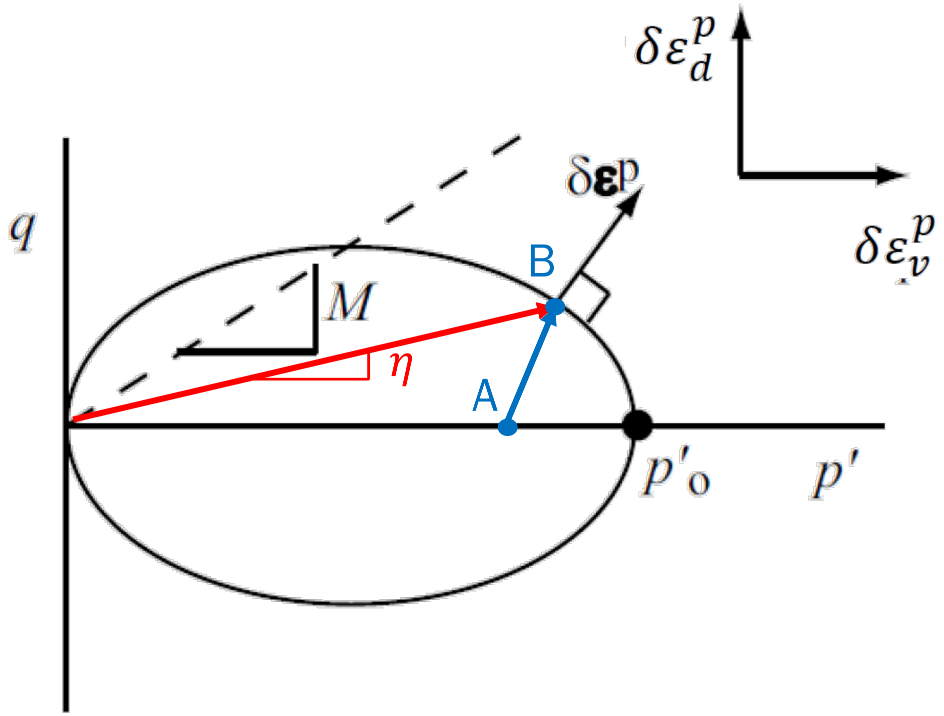
Volumetric and deviatoric plastic strain

$$d\varepsilon_v^p = \mu \frac{\partial g}{\partial p'}$$

$$d\varepsilon_d^p = \mu \frac{\partial g}{\partial q}$$

MCC - Plastic strains increments

The plastic strain increments are given as usual by the partial derivatives of the plastic potential function



$$\begin{pmatrix}\delta\varepsilon_v^p\\ \delta\varepsilon_d^p\end{pmatrix} = \mu\begin{pmatrix}\dfrac{\partial g}{\partial p'}\\ \dfrac{\partial g}{\partial q}\end{pmatrix} = \mu\begin{pmatrix}2p' - p'_0\\ \dfrac{2q}{M^2}\end{pmatrix}$$

The ratio provides the direction of the plastic strains:

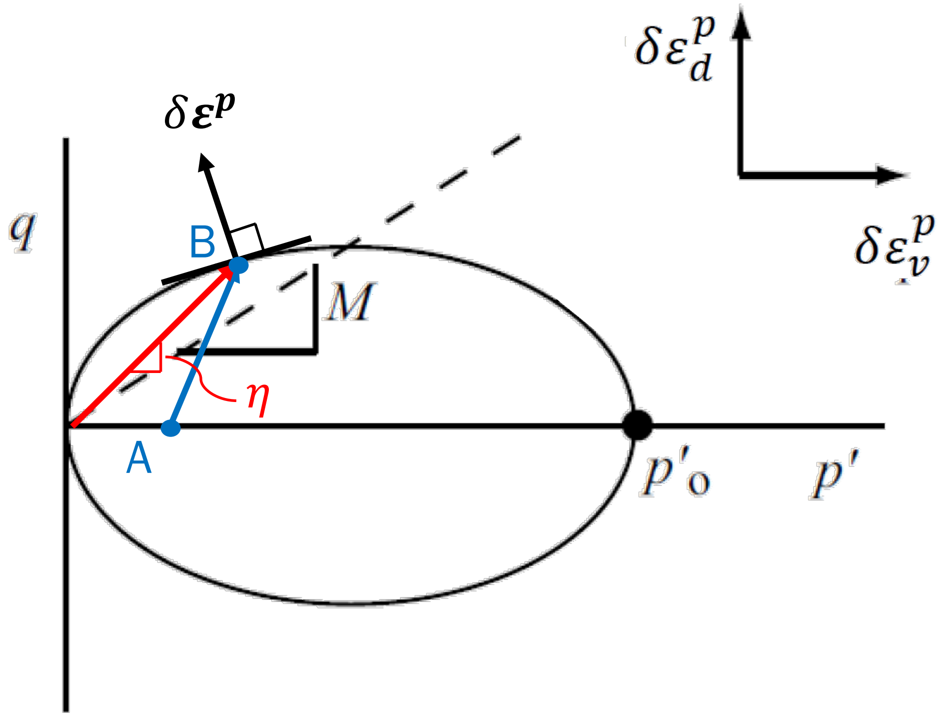
$$\frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} = \frac{M^2 - \eta^2}{2\eta}$$

$$\eta = \frac{q}{p'} < M \Longrightarrow \frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} > 0$$

<u>Compression plus distorsion</u>

MCC - Plastic strains increments

The plastic strain increments are given as usual by the partial derivatives of the plastic potential function



$$\begin{pmatrix} \delta\varepsilon_v^p \\ \delta\varepsilon_d^p \end{pmatrix} = \mu \begin{pmatrix} \dfrac{\partial g}{\partial p'} \\ \dfrac{\partial g}{\partial q} \end{pmatrix} = \mu \begin{pmatrix} 2p' - p'_0 \\ \dfrac{2q}{M^2} \end{pmatrix}$$

The ratio provides the direction of the plastic strains:

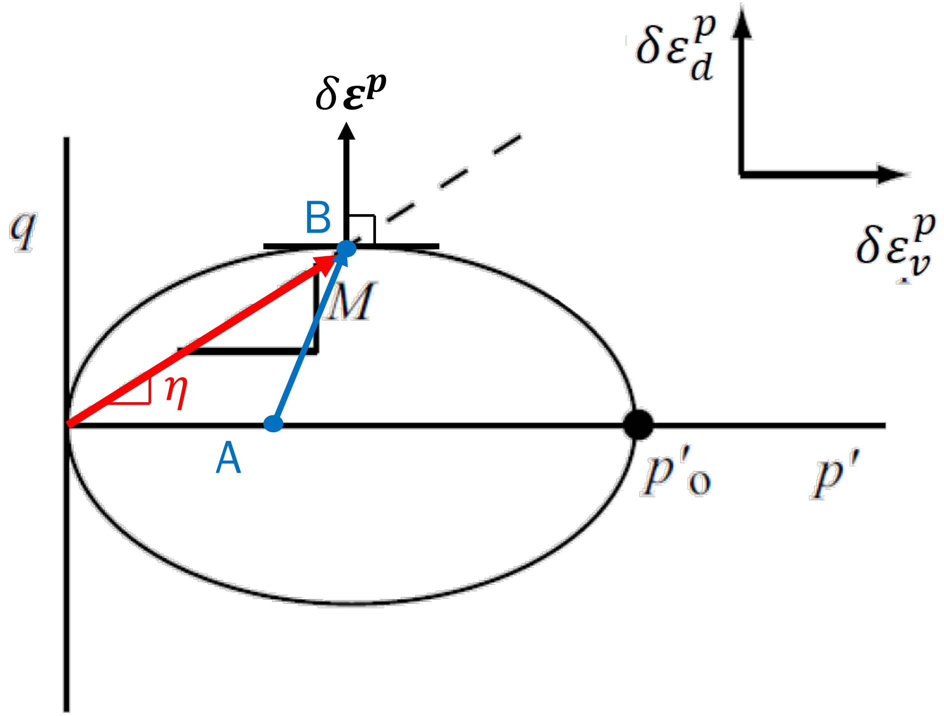
$$\frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} = \frac{M^2 - \eta^2}{2\eta}$$

$$\eta > M \Longrightarrow \frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} < 0$$

Expansion (dilation) plus distorsion

MCC - Plastic strains increments

The plastic strain increments are given as usual by the partial derivatives of the plastic potential function



$$\begin{pmatrix}\delta\varepsilon_v^p\\ \delta\varepsilon_d^p\end{pmatrix} = \mu\begin{pmatrix}\dfrac{\partial g}{\partial p'}\\ \dfrac{\partial g}{\partial q}\end{pmatrix} = \mu\begin{pmatrix}2p' - p'_0\\ \dfrac{2q}{M^2}\end{pmatrix}$$

The ratio provides the direction of the plastic strains:

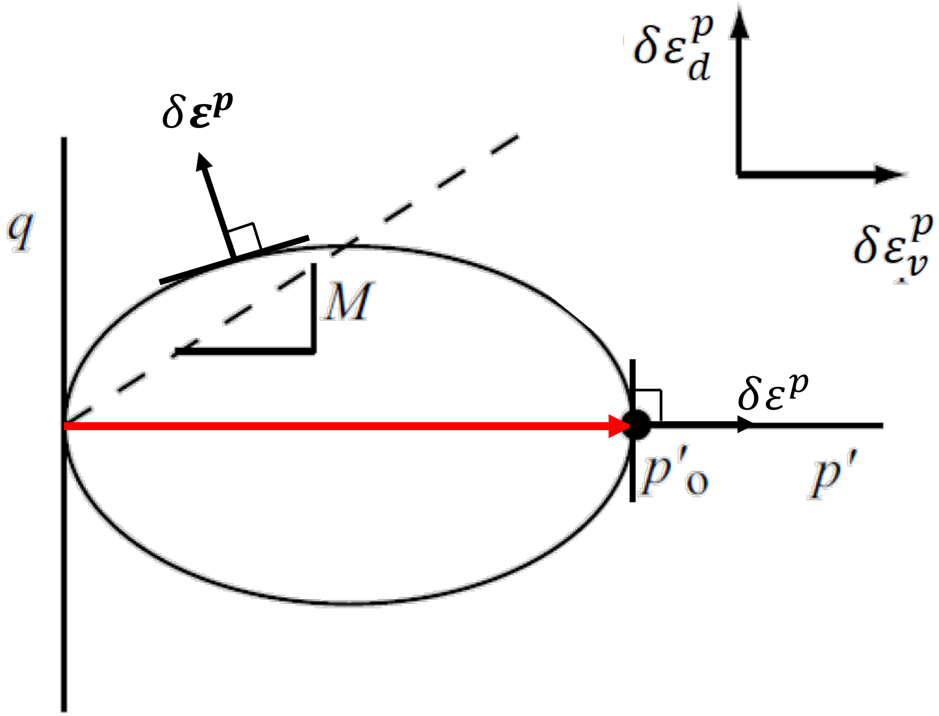
$$\frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} = \frac{M^2 - \eta^2}{2\eta}$$

$$\eta = M \Longrightarrow \frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} = 0$$

<u>Distorsion without compression</u> (critical state condition)

MCC - Plastic strains increments

The plastic strain increments are given as usual by the partial derivatives of the plastic potential function



$$\begin{pmatrix}\delta\varepsilon_v^p\\ \delta\varepsilon_d^p\end{pmatrix} = \mu\begin{pmatrix}\dfrac{\partial g}{\partial p'}\\ \dfrac{\partial g}{\partial q}\end{pmatrix} = \mu\begin{pmatrix}2p' - p'_0\\ \dfrac{2q}{M^2}\end{pmatrix}$$

The ratio provides the direction of the plastic strains:

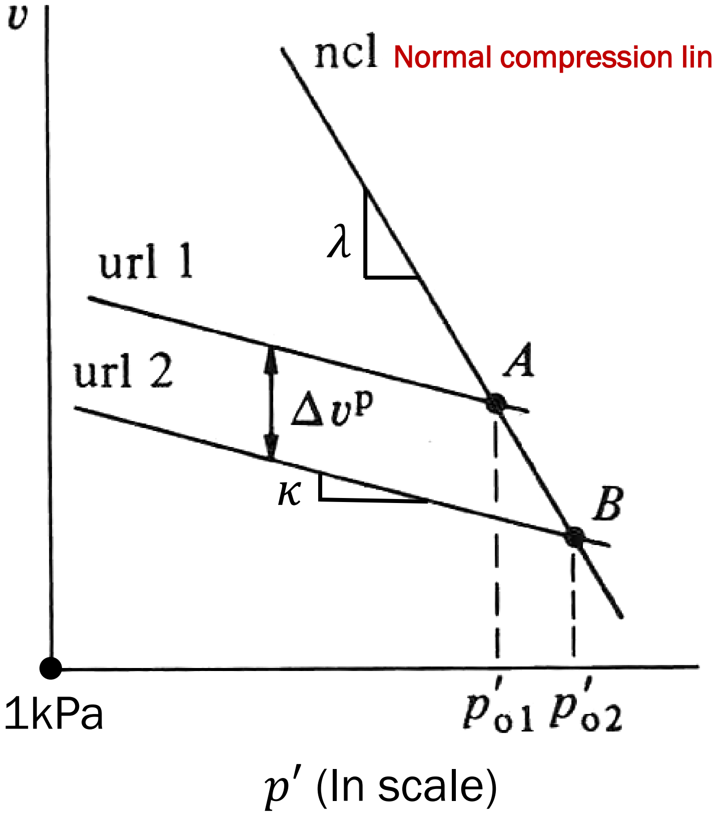
$$\frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} = \frac{M^2 - \eta^2}{2\eta}$$

$$\eta = 0 \Longrightarrow \frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} \to \infty$$

<u>Compression without distortion</u> (isotropic compression)

MCC – Normal compression line

Successive unloading-reloading paths



Wood, 1990

Equation of the normal compression line:

$$v = N - \lambda \ln \frac{p'}{p'_{ref}}$$

N is the specific volume at the reference mean effective stress (p'_{ref})

If $p' = p'_{ref} \rightarrow v \Rightarrow N$

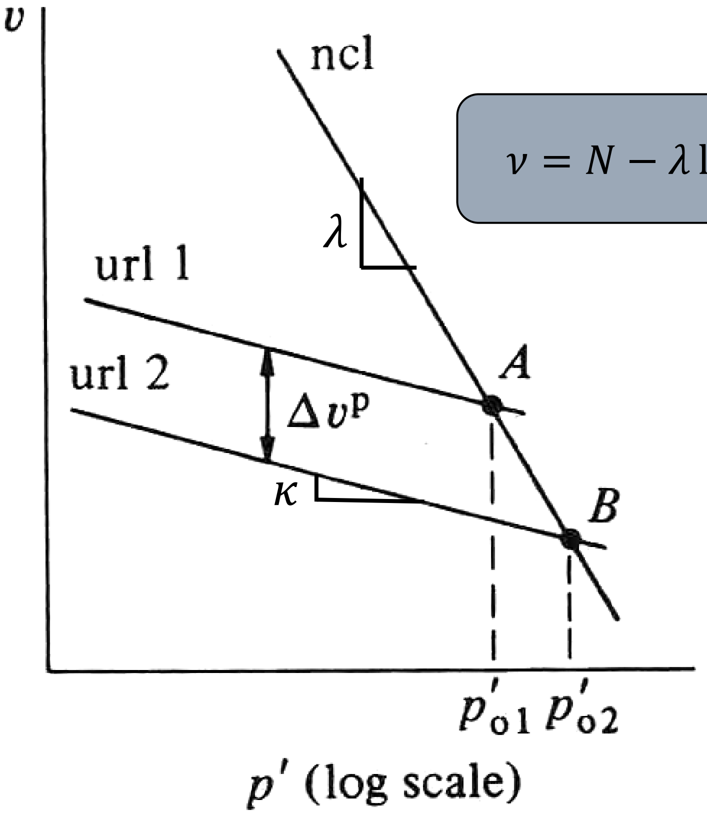
Tipically it is assumed: $p'_{ref} = 1\ kPa$, so the expression becomes:

$$v = N - \lambda \ln p'$$

But keep in mind the units (e.g. kPa)!!

MCC – Hardening rule

Successive unloading-reloading paths



$$v = N - \lambda \ln p'$$

Wood, 1990

$$\Delta v^p = \Delta v - \Delta v^e = -(\lambda - \kappa) \ln \left(\frac{p'_{02}}{p'_{01}} \right)$$

$$dv^p = -(\lambda - \kappa) \frac{dp'_0}{p'_0}$$

$$d\varepsilon_v^p = \frac{(\lambda - \kappa)}{v} \frac{dp'_0}{p'_0}$$

$$dp'_0 = \frac{v p'_0}{(\lambda - \kappa)} d\varepsilon_v^p$$

MCC – Summary

Elastic part $d\varepsilon_v^e = \frac{\kappa}{vp'}dp'$ $d\varepsilon_d^e = \frac{dq}{3G}$

Yield function $F = q^2 - M^2[p'(p_0' - p')] = 0$

Plastic potential $g = F = q^2 - M^2[p'(p_0' - p')] = 0$

Hardening rule $\frac{dp_0'}{p_0'} = \frac{v}{\lambda - \kappa}d\varepsilon_v^p$

MCC parameters

- Elastic: κ, G
- Plastic: M, λ, N

MCC – Summary

Full plastic compliance relationship

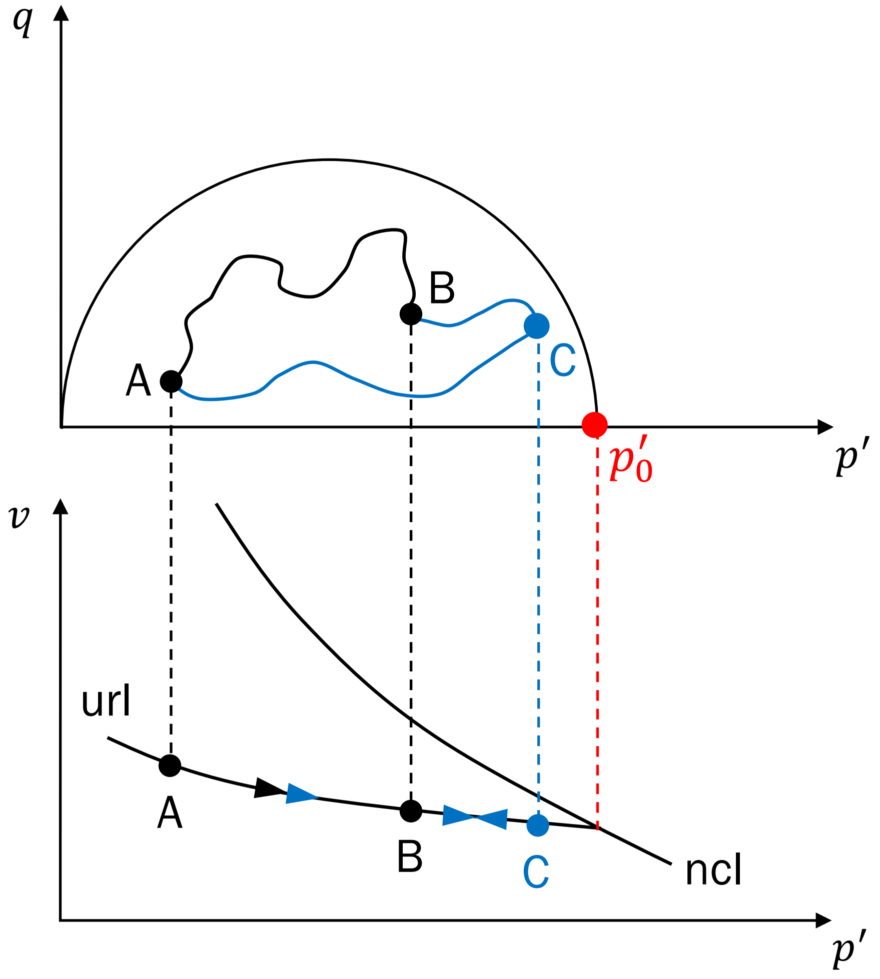
$$\begin{pmatrix}\delta\varepsilon_v^p\\ \delta\varepsilon_d^p\end{pmatrix} = \frac{\lambda-\kappa}{vp'(M^2+\eta^2)}\begin{pmatrix} M^2-\eta^2 & 2\eta \\ 2\eta & \frac{4\eta^2}{M^2-\eta^2}\end{pmatrix}\begin{pmatrix}\delta p'\\ \delta q\end{pmatrix}$$

Full elasto-plastic stiffness relationship

$$\begin{pmatrix}\delta p'\\ \delta q\end{pmatrix} = \left[\begin{pmatrix}\frac{vp'}{\kappa} & 0\\ 0 & 3G\end{pmatrix} - \frac{\begin{pmatrix}\left(\frac{vp'}{\kappa}\right)^2(2p'-p_0')^2 & \frac{6Gvp'q(2p'-p_0')}{M^2\kappa}\\ \frac{6Gvp'q(2p'-p_0')}{M^2\kappa} & \frac{36G^2q^2}{M^4}\end{pmatrix}}{\frac{vp'}{\kappa}(2p'-p_0')^2+\frac{12Gq^2}{M^4}+\frac{vp'p_0'(2p'-p_0')}{\lambda-\kappa}}\right]\begin{pmatrix}\delta\varepsilon_v\\ \delta\varepsilon_d\end{pmatrix}$$

MCC Behaviours

MCC – Elastic behaviour



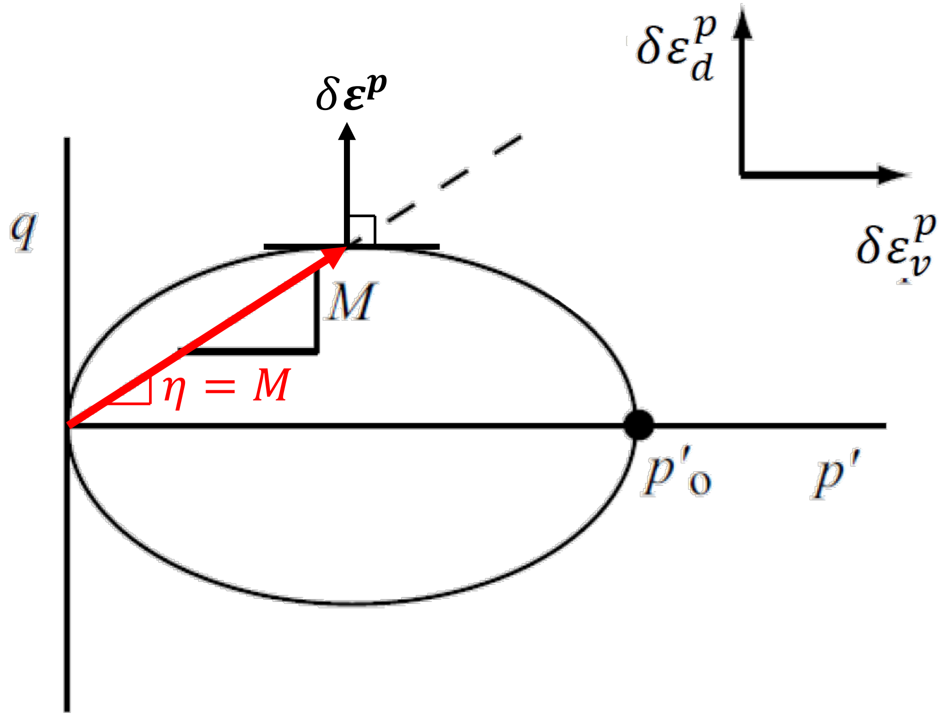
- The elastic volumetric behaviour occurs along the unloading-reloading line (url).
- Each url is univocally associated with the current preconsolidation pressure p'_0
- For all the stress-paths occuring inside the yield surface, the specific volume changes following the url; its final value depends only on the final value of p', independently from the stress-path

$$\boxed{d\varepsilon_v^e = \frac{\kappa}{1+e}\frac{dp'}{p'}}$$

MCC - Plastic strains increments

What happens as η tends to M?

- The increments of plastic volumetric strains becomes smaller and smaller
- As a consequence, the hardening behaviour tends to stop
- The shear compliance tends to infinity (shear stiffness tends to zero)
- Asymptotically, perfectly plastic condition are reaches, where shear strain continues without any change in yield locus size, stresses and volumetric strain



→ Critical state

$$\eta \rightarrow M \Longrightarrow \delta\varepsilon_v^p \rightarrow 0 \Longrightarrow \delta {p_0}' \rightarrow 0$$

$$\eta \rightarrow M \Longrightarrow \frac{\delta\varepsilon_d^p}{\delta q} \rightarrow \infty$$

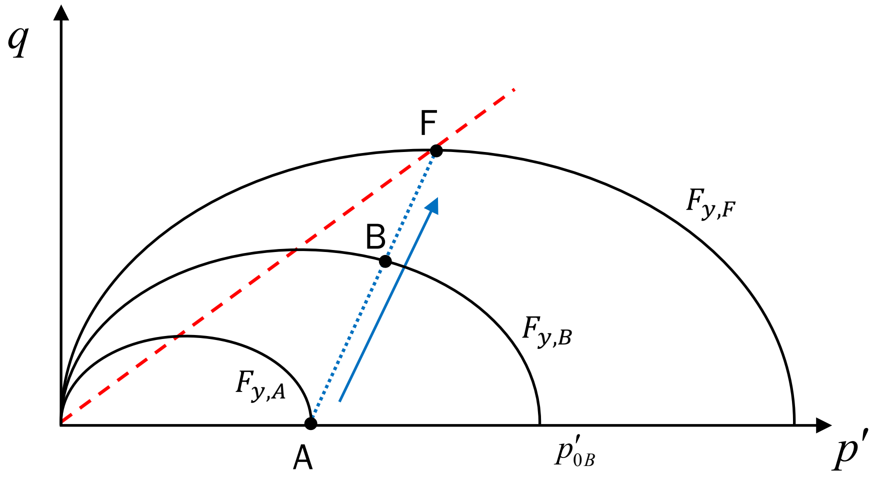
Tends to zero

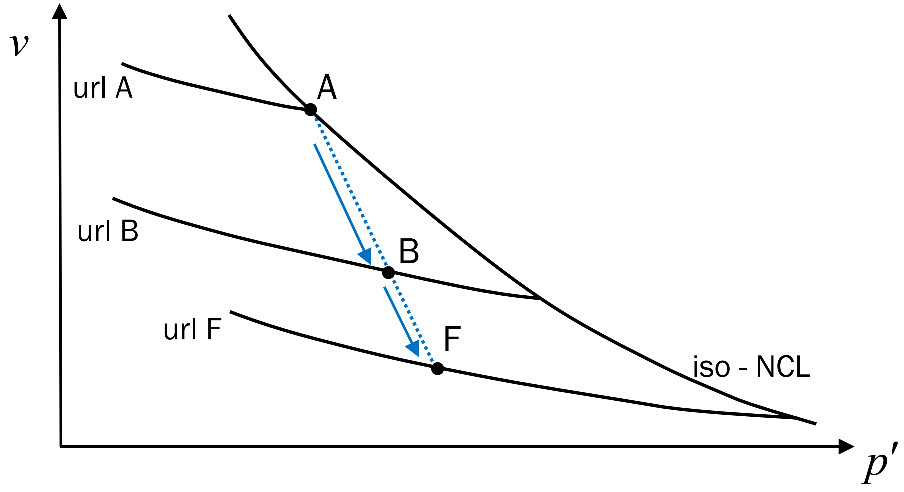
$$\begin{pmatrix}\delta\varepsilon_v^p \\ \delta\varepsilon_d^p\end{pmatrix} = \frac{\lambda-\kappa}{vp'(M^2+\eta^2)}\begin{pmatrix} M^2-\eta^2 & 2\eta \\ 2\eta & \dfrac{4\eta^2}{M^2-\eta^2}\end{pmatrix}\begin{pmatrix}\delta p' \\ \delta q\end{pmatrix}$$

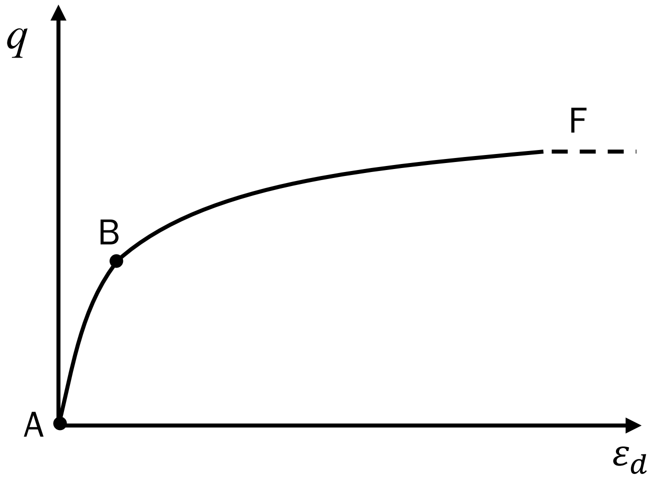
Tends to infinity

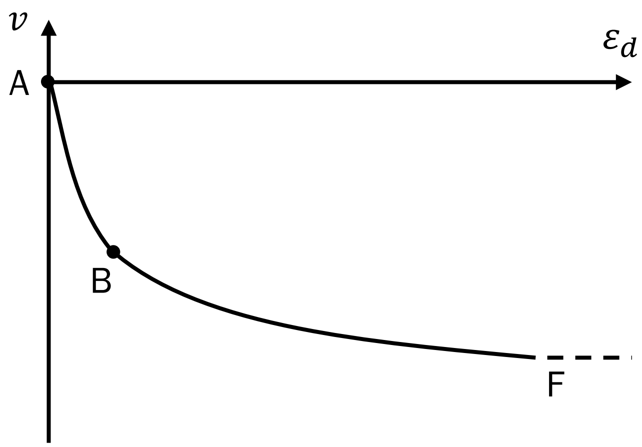
MCC – Prediction

Drained conventional triaxial compression test – Normally consolidated





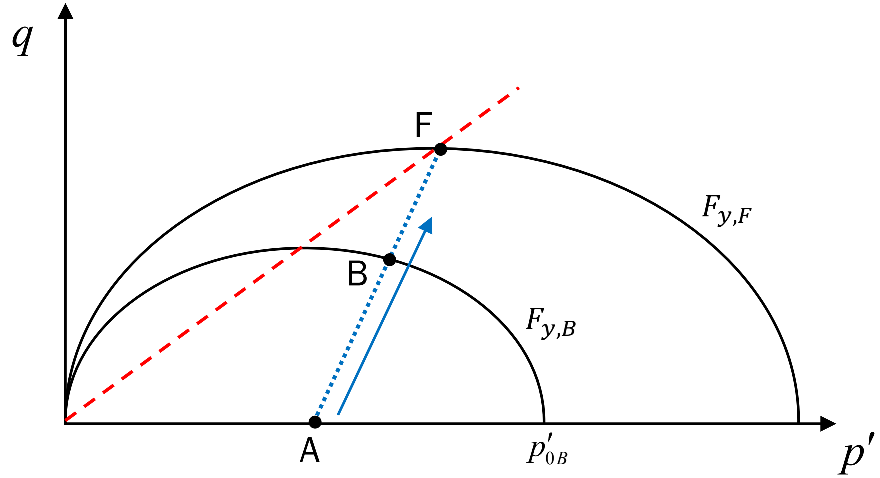


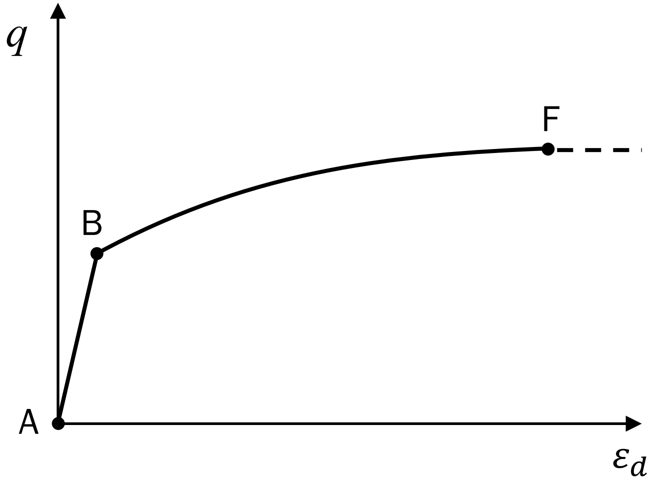


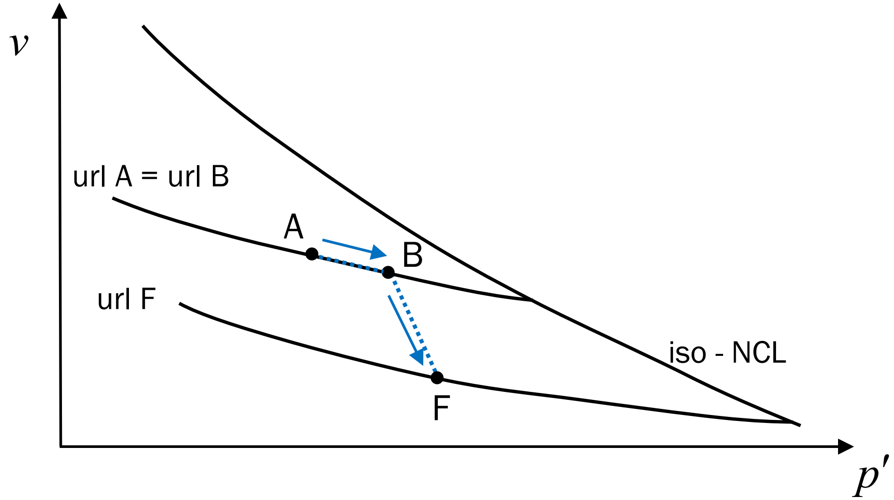
Wood, 1990

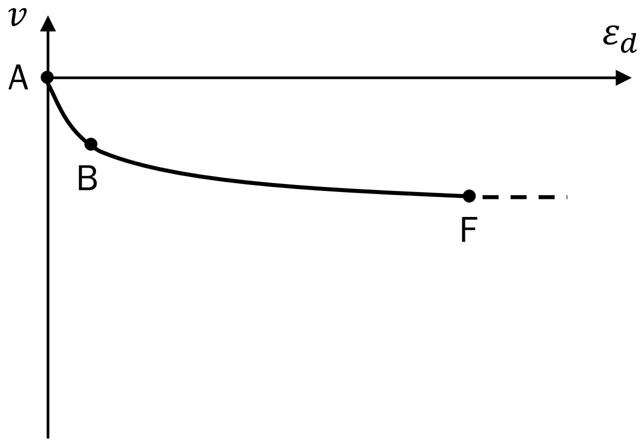
MCC – Prediction

Drained conventional triaxial compression test – Lightly over consolidated





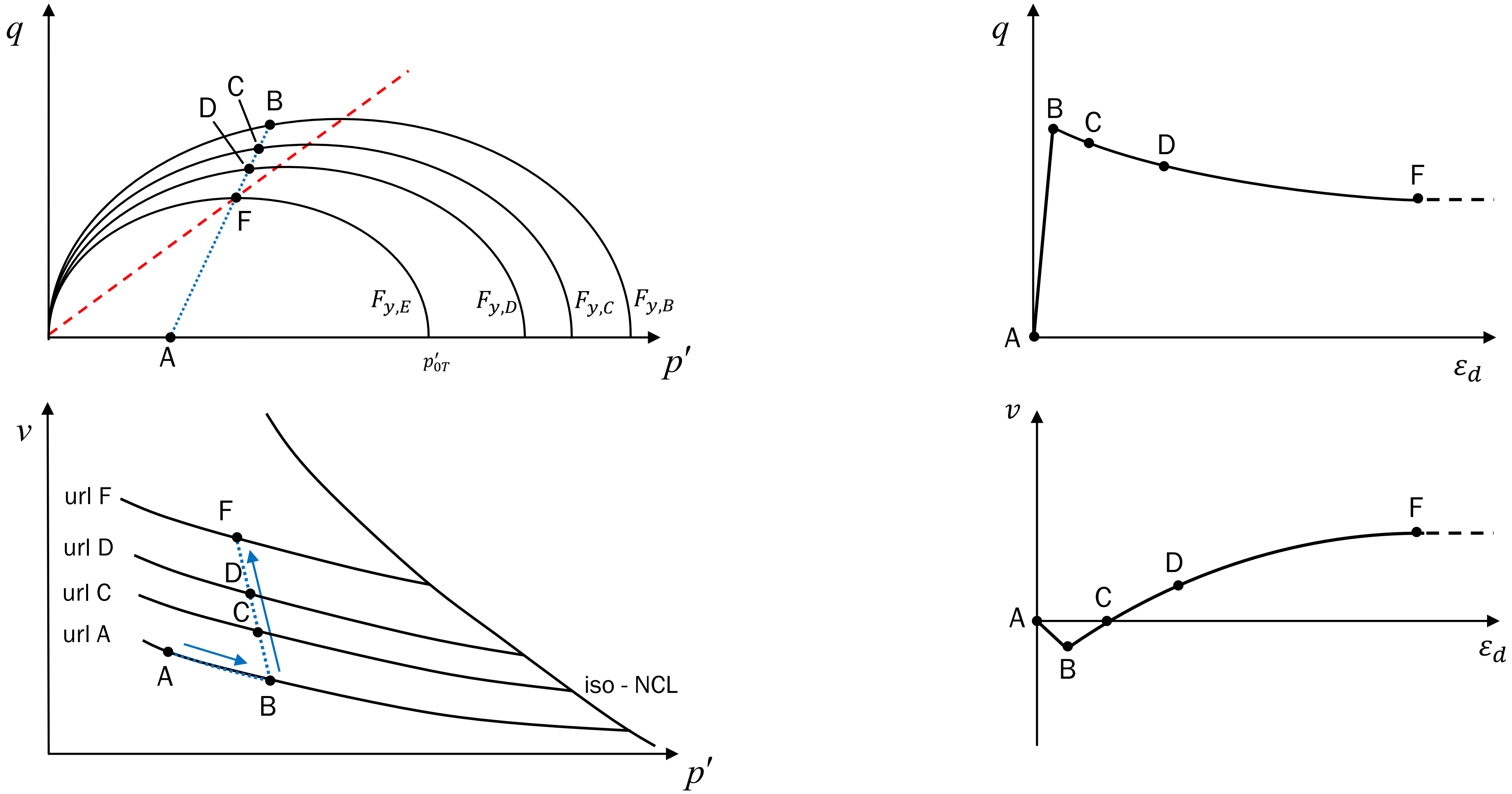




Wood, 1990

MCC – Prediction

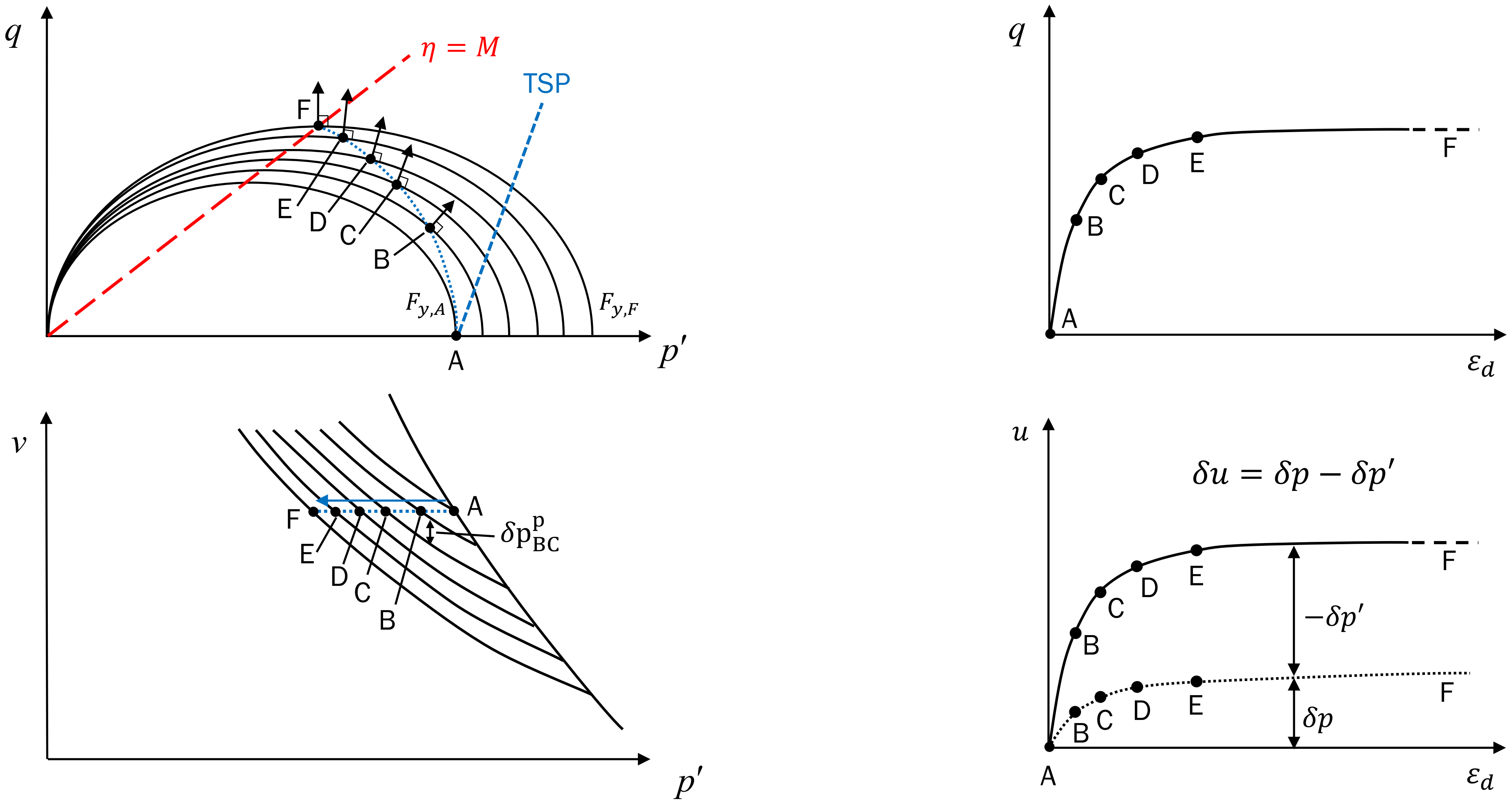
Drained conventional triaxial compression test – Heavily over consolidated



Wood, 1990

MCC – Prediction

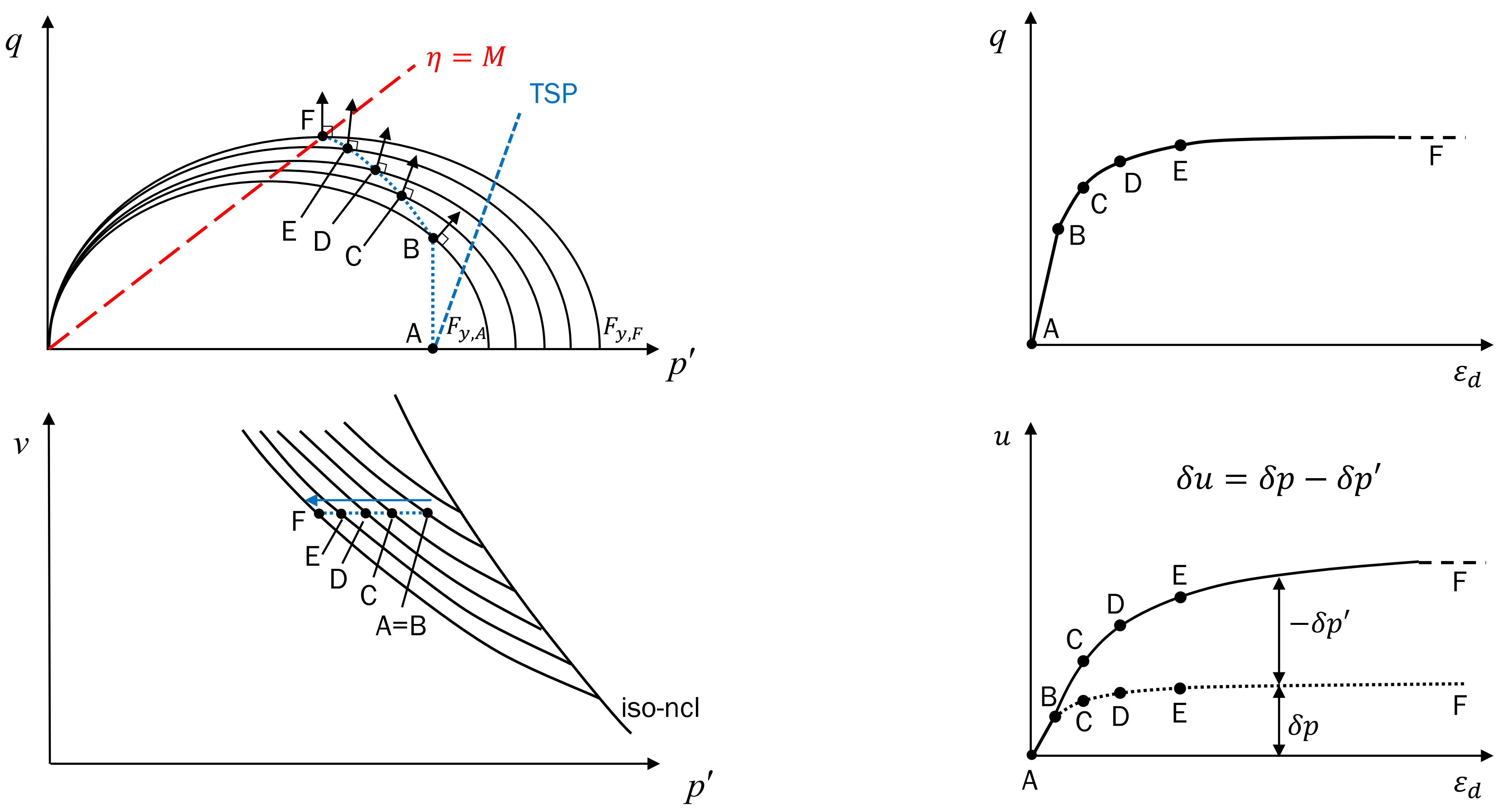
Undrained conventional triaxial compression test – Normally consolidated



Wood, 1990

MCC – Prediction

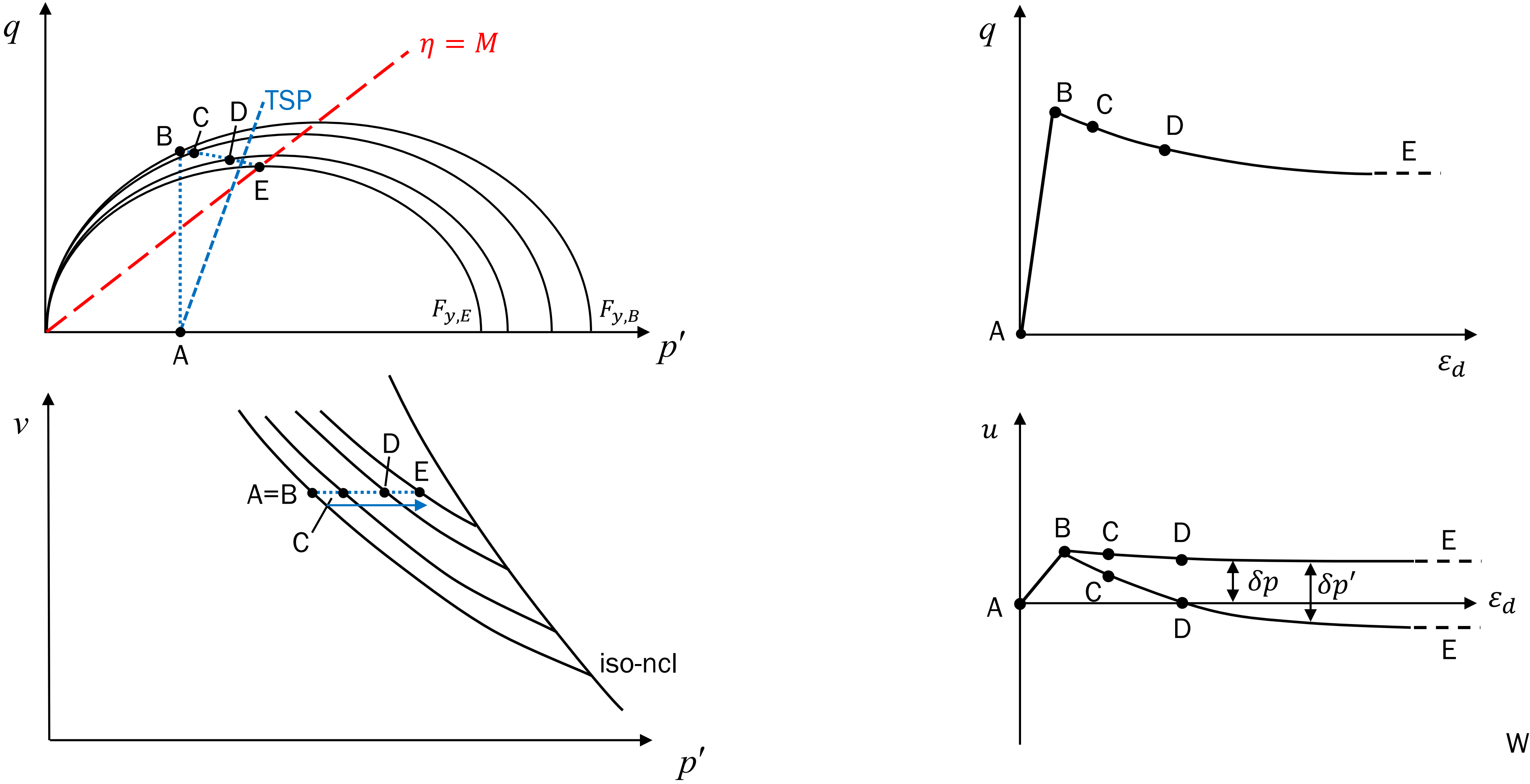
Undrained conventional triaxial compression test - Lightly over consolidated



Wood, 1990

MCC – Prediction

Undrained conventional triaxial compression test - Heavily over consolidated



Wood, 1990

Conclusion

Conclusion

Critical state :

State in which the soil reaches its ultimate shear strength; the shearing can continue without any tendency of the soil to change its volume

Modified Cam-Clay :

Modelling of soil behavior within the framework of hardening elasto-plasticity

Estimation of elastic and plastic deformation using MCC

MCC can be used for hand calculations of elements undergoing simple modes of deformation

References

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4. Wood, D. M. (1990). Soil behavior and critical state soil mechanics. Cambridge University Press, Cambridge.
5. Yu, H. S. (2006). Plasticity and Geotechnics. Advances in Mechanics and Mathematics. Springer, New York.
6. Atkinson, J. H. and Bransby, P. L. (1978). The Mechanics of Soils, An Introduction to Critical State Soil Mechanics. McGraw-Hill, London.

Thank you for your attention